J. Appl. Maths Mechs Vol. 55, No. 5, pp. 714–723, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00

FUNDAMENTAL SOLUTIONS IN AN ELASTIC SPACE IN THE CASE OF MOVING LOADS†

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(*Received* 3 *May* 1990)

Green's matrix is constructed for a point force moving along an axis in an isotropic elastic space at sub-, super- and transonic speeds. The theory of generalized functions is used to establish an analogue of Somigliana's formula which enables one to construct boundary integral equations (BIEs) for solving boundary-value problems with moving loads.

THE STUDY of transportation in tunnels conveying loads and fluids in conduits leads to model problems in which one has to deal with the effect of moving loads in cylindrical cavities in continuous media. In the classical case of cavities with circular profiles, such problems can be successfully tackled by means of complete or incomplete separation of variables [1]. For more-complicated profiles the method of BIEs is more convenient. The point of departure in this method is the existence of fundamental solutions. Fundamental solutions for subsonic velocities have been known for some time [2, 3]. This paper proposes a new, simpler way of constructing fundamental solutions, valid in any velocity range; the theory of generalized functions is used to construct BIEs for solving problems involving the effect of moving loads in tunnels of arbitrary cross-section.

1. THE EQUATIONS OF MOTION IN GENERALIZED FUNCTION SPACE

Let x_1, x_2, x_3 denote Lagrangian Cartesian coordinates of a point x in a linearity-elastic isotropic medium specified in terms of its Lamé parameters λ, μ and the density ρ; u_1, ε_{ij}, σ_{ij} will denote the Cartesian coordinates of the displacements $\mathbf{u}$, and the strain and stress tensors, respectively. The relationship between these quantities is governed by the Cauchy relations and Hooke's law:

$$\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i}),\quad \sigma_{ij} = \lambda u_{k,k}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{1.1}$$

Throughout, the repeated-index summation convention will be used; δ_{ij} will denote the Kronecker delta, and $u_{i,j} = \partial u_i/\partial x_j$, $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$.

Using (1.1), we can reduce the transport equations of the continuous medium

$$\sigma_{ij,j} + \rho G_i = \rho u_{i,tt},\quad i, j = 1, 2, 3$$

†*Prikl. Mat. Mekh.* Vol. 55, No. 5, pp. 840–848, 1991.

to the form

$$(c_1^2 - c_2^2)u_{j,ij} + c_2^2 u_{i,jj} - u_{i,tt} + G_i = 0 \tag{1.2}$$

where c_1, c_2 are the velocities of the body and shear waves. If $G_i = G_i(x_1, x_2, x_3 - ct)$, then the solution of Eq. (1.2) has the same structure, $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3 - ct)$. In a moving coordinate frame $(x_1', x_2', x_3') = (x_1, x_2, x_3 - ct)$, Eq. (1.2) becomes (with the prime omitted)

$$(c_1^2 - c_2^2)u_{j,ij} + c_2^2 u_{i,jj} - c^2 u_{i,33} + G_i = 0 \tag{1.3}$$

As system (1.2) is strictly hyperbolic, it may have discontinuous solutions [4]. The surface of discontinuity (wave front) is a characteristic surface of system (1.2) and propagates through space with time. Let $F(\mathbf{x}, t) = 0$ be the equation of such a surface F_t, $\mathbf{n} = (n_1, n_2, n_3)$ the unit vector normal to F_t in R_3:

$$n_j = F_{,j} \| \operatorname{grad} F_t \|^{-1}, \quad \| \operatorname{grad} F_t \|^2 = \sum_{k=1}^{3} \left(\frac{\partial F_t}{\partial x_k} \right)^2$$

v the velocity of F_t in R_3:

$$v = -F_{,t} \| \operatorname{grad} F_t \|^{-1} \tag{1.4}$$

Let F denote "the same" surface, but in $R_4 = R_3 \times t$, where it is at rest $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_4)$ the normal to F in R_4:

$$\nu_j = F_{,j} \| \operatorname{grad} F \|^{-1}, \quad \| \operatorname{grad} F \|^2 = \sum_{k=1}^{4} \left(\frac{\partial F}{\partial x_k} \right)^2$$

The surface F is defined by the equation

$$\nu_t = \pm c_j \sqrt{\sum_{k=1}^{3} \nu_k^2}, \quad j = 1, 2, \ \nu_t = \nu_4 \tag{1.5}$$

It follows from (1.5) and (1.4) that v is one of the sonic velocities c_1, c_2.

The requirement that the displacements remain continuous across the front, which is necessary in order to maintain the continuity of the medium

$$[u_i]_{F_t} = 0 \tag{1.6}$$

leads to well-known kinematic compatibility conditions for the solutions [4]:

$$[u_{i,t} n_j + v u_{i,j}]_{F_t} = 0 \tag{1.7}$$

(the continuity of the tangential derivatives), and in addition Eqs (1.2) imply the dynamical compatibility conditions

$$[\sigma_{ij} n_j + \rho v u_{i,t}]_{F_t} = 0 \tag{1.8}$$

Here $[f]_{F_t}$ denotes the jump in f across F_t: $[f_i n_i]_{F_t} \triangleq n_i [f_i]_{F_t}$.

Let us see how these conditions transform for solutions of Eqs (1.3). The characteristic surface T must satisfy the equation

$$\det \{ (c_1^2 - c_2^2) n_i n_j + \delta_{ij} (c_2^2 - c^2 n_3^2) \} = 0 \tag{1.9}$$

whose roots, by (1.5), are

$$n_3 = \pm M_j^{-1} = \pm c_j/c, \ j = 1, 2 \tag{1.10}$$

Here $\mathbf{n} = \{n_i\}$ is the unit normal vector to T and M_j are the Mach numbers.

Since $n_3 \leqslant 1$, this implies that $M_j \geqslant 1$, i.e., discontinuous solutions may occur only if the loads are supersonic or sonic. In that case the conditions at the fronts are

$$[u]_T = 0, \ [vu_{i,j} - cn_j u_{i,3}]_T = 0 \tag{1.11}$$

$$[\sigma_{ij} n_j - \rho v c u_{i,3}]_T = 0 \tag{1.12}$$

By (1.10) and (1.4), $v = cn_3$; substituting this into (1.12), we get

$$[\sigma_{ij} n_j - \rho c^2 n_3 u_{i,3}]_T = 0 \tag{1.13}$$

To construct fundamental solutions, we have to introduce singular body forces in Eqs (1.3); this, in turn, requires us to write the equations, with due attention to conditions (1.11) and (1.13), in the space of generalized functions. As the test function space $D_3(R_3)$ we take the space of compactly-supported infinitely differentiable vector-values functions $\varphi(x) = \{\varphi_1(x), \ldots, \varphi_3(x)\}$ defined on R_3. The dual is the space of generalized vector-valued functions $D_3'(R_3)$; henceforth we shall say simply "function" for "vector-valued function". Convergence is defined in the same way as convergence in $D(R_N)$ and $D'(R_N)$ [5, 6].

Let $\mathbf{u}(\mathbf{x})$ be any classical solution of (1.3) which is continuous and twice piecewise differentiable almost everywhere, except perhaps on the surfaces where conditions (1.11)–(1.13) are satisfied. Let $\mathbf{u}^*(\mathbf{x})$ denote the generalized function corresponding to $\mathbf{u}(\mathbf{x})$: $u^* = u$, i.e.

$$(\mathbf{u}^*, \boldsymbol{\varphi}) = \int_{R_3} u_i(\mathbf{x})\, \varphi_i(\mathbf{x})\, dV, \quad \forall \varphi \in D_3(R_3) \tag{1.14}$$

The integration is performed over R_3, or more precisely, over part of it, as $\boldsymbol{\varphi}(\mathbf{x})$ has compact support.

We will now introduce the generalized stress and strain tensors σ_{ij}^* and ε_{ij}^*, respectively, defined by Eqs (1.1) but now in the generalized sense.

The characteristic function of the set $T_+ = \{\mathbf{x}: T(\mathbf{x}) > 0\}$ (where $T(\mathbf{x}) = 0$ is the equation of the surface T) is defined as follows:

$$H_T^+(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > 0 \\ {}^1/_2, & T(\mathbf{x}) = 0 \\ 0, & T(\mathbf{x}) < 0 \end{cases} \tag{1.15}$$

Similarly we define T_- and H_{T^-}: $H_{T^-} + H_{T^+} = 1$. It is known [5] that

$$H_{T,j}^{\pm} = \pm n_j \delta_T(\mathbf{x}) \tag{1.16}$$

$$u_{i,j}^* = u_{i,j} + [u_i n_j]_T \delta_T(x) \tag{1.17}$$

where $\boldsymbol{\alpha}(\mathbf{x})\, \delta_T(\mathbf{x})$ is a simple layer on T:

$$(\alpha \delta_T, \boldsymbol{\varphi}) = \int_T \alpha_i(\mathbf{x})\, \varphi_i(\mathbf{x})\, dS, \quad \forall \varphi \in D_3(R_3) \tag{1.18}$$

The first term on the right in (1.17) is the classical derivative of u_i.

It follows from these relationships that

$$\sigma^*_{ij,j} - \rho c^2 u^*_{i,33} + \rho G_i^* = [\sigma_{ij}n_j - \rho c^2 n_3 u_{i,3}]_T \delta_T + \{[\lambda u_k n_k \delta_{ij} + \mu (u_i n_j + u_j n_i)]_T \delta_T\}_{,j} - \{[u_k n_3]_T \delta_T\}_{,3} \quad (1.19)$$

It is obvious that the density $\alpha(\mathbf{x})$ of the simple layer is identical with the dynamical compatibility conditions for solutions of (1.13) and vanishes. Conditions (1.11) imply that the density $\beta(x)$ of the double layer $\{\beta(x)\delta_T(x)\}_{,j}$ also vanishes. Thus the right side of (1.19) vanishes. Thus the displacements expressed in terms of generalized functions satisfy the same equations (1.3), but now in the generalized sense.

2. THE SOMIGLIANA FORMULA FOR MOVING LOADS

Let $U_{ij}^*(\mathbf{x})$ be a fundamental solution of Eqs (1.3) corresponding to a body force $\mathbf{G}^* = \delta_{ij}\delta(\mathbf{x})$, where $\delta(\mathbf{x})$ is a delta-function in $D'(R_3)$ [6]:

$$(c_1^2 - c_2^2) U^*_{jk,ij} + c_2^2 U^*_{ik,jj} - c^2 U^*_{ik,33} + \delta_{ik}\delta(\mathbf{x}) = 0 \quad (2.1)$$

We define tensors corresponding to $G_i^* = \delta_{ij}\delta(\mathbf{x}-\mathbf{y})$

$$U_{kj}(\mathbf{x}, \mathbf{y}) = U_{kj}^*(\mathbf{x} - \mathbf{y})$$

$$\Gamma_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{n}) = \lambda n_i U_{kj,k} + \mu n_k (U_{ij,k} + U_{kj,i})$$

$$T_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{n}) = \Gamma_{ji}(\mathbf{y}, \mathbf{x}, \mathbf{n}) \quad (2.2)$$

Now it is well known [5, 6] that the displacements corresponding to any given body force $\mathbf{G}^*$ are determined by the following convolution:

$$\mathbf{u}^* = U_{ij}^* * G_j^* = \int_{R_3} U_{ij}(\mathbf{x}, \mathbf{y}) G_j^*(\mathbf{y})\, dV(\mathbf{y}) \quad (2.3)$$

where the last equality holds for regular U_{ij}^* and G_i^*.

Let S_+ be an elastic medium, bounded by a smooth cylindrical surface S, along which the load is moving in the direction of the x_3 axis at a constant velocity c:

$$\sigma_{ij}n_j = p_i(x_1, x_2, x_3 - ct), \quad \mathbf{x} \in S \quad (2.4)$$

where $\mathbf{n} = \{n_i\}$ is the outward unit normal vector to S. Clearly,

$$n_3 = 0 \quad (2.5)$$

There are no body forces. Let $\mathbf{u}(\mathbf{x})$ be a solution of Eqs (1.3) in the moving coordinate frame, satisfying (2.4) in S_+. Consider the generalized function $\mathbf{u}^*(\mathbf{x}) = \mathbf{u}(\mathbf{x})H_{S^+}(\mathbf{x})$. Following the reasoning of Sec. 1, we obtain the following equations for this function, analogous to (1.19):

$$\sigma^*_{ij,j} - \rho c^2 u^*_{i,33} = -(p_i - \rho c^2 n_3 u_{i,3})\delta_S(\mathbf{x}) + \{n_3 u_i \delta_S(\mathbf{x})\}_{,3} - \{(\lambda u_k n_k \delta_{ij} + \mu (u_i n_j + u_j n_i))\delta_S(\mathbf{x})\}_{,j} \quad (2.6)$$

where the jump is now replaced by the values of the expressions on S, because $\mathbf{u}^* = 0$ outside $S_+ + S$. Following (2.3) and using (2.5) and the rules for the differentiation of convolutions [5], we obtain

$$\rho u_i^* = U_{ik}^* * p_k\delta_S + \{(\lambda u_k n_k \delta_{lj} + \mu (u_l n_j + u_j n_l))\delta_S * U^*_{il}\}_{,j} \quad (2.7)$$

or

$$\rho u_i^* = U_{ik}^* * p_k\delta_S + (\lambda u_k n_k \delta_{lj} + \mu (u_l n_j + u_j n_l))\delta_S * U_{il,j}^* \quad (2.8)$$

If U_{ik}^* and $U_{ik,j}^*$ are regular generalized functions, the last equality may be written in integral notation. However, it has been shown that, depending on the velocity of motion c, the tensor U_{ik}^* or its derivatives may become singular. For such U_{ik}^* it is convenient to use (2.7). In particular, if U_{ik}^* is regular, then all the convolutions in (2.7) can be written in integral notation and the integrals then differentiated by the usual rules.

We will use the following corollary of (2.1):

$$U_{ij}(\mathbf{x}, \mathbf{y}) = U_{ij}(\mathbf{y}, \mathbf{x}) = U_{ji}(\mathbf{x}, \mathbf{y})$$

and the equality

$$U_{ij, x_k} = -U_{ij, y_k}$$

to rewrite (2.8) formally in integral notation, transforming the dummy indices with the aid of (2.2). The result is

$$u_i^* = \int_S \{U_{ik}(\mathbf{x}, \mathbf{y})\, p_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y}, \mathbf{n}(\mathbf{y}))\, u_k(\mathbf{y})\}\, dS(\mathbf{y}) \quad (2.9)$$

This formula, formally the same as the Somigliana formula of static elasticity theory [3], expresses the displacements in the interior of the domain in terms of the boundary values of the displacements and the stresses. For Lyapunov surfaces, if $x \in S$ the formula gives a BIE for solving boundary-value problems with moving loads. In that case, however, the integrals must be regularized in some way, depending on the properties of the kernels U_{ij} and T_{ij}. The latter depend essentially on the velocity c. It has been shown that (2.9) as it stands is true only at subsonic velocities of the moving load ($c < c_2$), while for $x \in S$ the second integral is evaluated as a singular integral in the principal-value sense. For supersonic velocities ($c \geqslant c_2$) one uses (2.7) to construct a regular analogue of the Somigliana formulae.

3. THE GENERALIZED FOURIER TRANSFORM

We will first describe some constructive algorithms for computing Fourier and inverse Fourier transforms of locally integrable slowly increasing functions in R_1 [5], which we will then use to determine U_{ij}^*.

Let $f(x)$ be a locally integrable slowly-increasing function in R_1. It Fourier transform is defined as

$$F_x[f] = \bar{f}(\xi) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(x)\, e^{i\xi x - \varepsilon|x|}\, dx, \quad \varepsilon > 0 \quad (3.1)$$

It is convenient to use the representation

$$f(x) = f_+(x) + f_-(x) \quad (3.2)$$

where $f_\pm = fH(\pm x)$, $H(x)$ being the Heaviside function. By (2.1),

$$\bar{f}(\xi) = \lim_{\varepsilon \to +0} (\bar{f}_+(\xi + i\varepsilon) + \bar{f}_-(\xi - i\varepsilon))$$

The limits are evaluated here in the sense of the convergence of generalized functions [5]. The integrals in (2.1) exist thanks to the slow increase in $f(x)$; moreover:

(a) $\bar{f}_+(\xi)$ is analytic if $\operatorname{Im}\xi>0$, $\bar{f}_+\to 0$ as $\operatorname{Im}\xi\to+\infty$,

(b) $\bar{f}_-(\xi)$ is analytic if $\operatorname{Im}\xi<0$, $\bar{f}_-\to 0$ as $\operatorname{Im}\xi\to-\infty$.

Since

$$\bar{f}_+(\xi+i\varepsilon)=F_x[f_+(x)\,e^{-\varepsilon x}],\quad \bar{f}_-(\xi-i\varepsilon)=F_x[f_-(x)e^{\varepsilon x}]$$

(in terms of the classical Fourier transform), we can use the inverse transform

$$f_+(x)\,e^{-\varepsilon x}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\bar{f}_+(\xi+i\varepsilon)\,e^{-i\xi x}\,d\xi=\frac{e^{-\varepsilon x}}{2\pi}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\bar{f}_+(\xi)\,e^{-i\xi x}\,d\xi$$

Hence

$$f_+(x)=\frac{1}{2\pi}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\bar{f}_+(\xi)\,e^{-i\xi x}\,d\xi,\quad \forall\varepsilon>0$$

A similar formula holds for $f_-(x)$.

Summing, we obtain

$$f(x)=\frac{1}{2\pi}\left(\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\bar{f}_+(\xi)\,e^{-i\xi x}\,d\xi+\int\limits_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\bar{f}_-(\xi)\,e^{-i\xi x}\,d\xi\right)\tag{3.3}$$

Obviously, this is also true for ordinary functions that have classical Fourier transforms. The Fourier transforms of locally integrable slowly-increasing functions have singularities on the real axis ξ' of the complex plane $\xi=\xi'+i\xi''$, so that formulae (3.3) can be conveniently used to determine the source function.

However, the available information usually includes $\bar{f}(\xi)$, rather than $\bar{f}_+$ and $\bar{f}_-$. If $\bar{f}(\xi)$ admits of the representation

$$\bar{f}(\xi)=\bar{f}^+(\xi)+\bar{f}^-(\xi)\tag{3.4}$$

where $\bar{f}_+$ and $\bar{f}_-$ have properties (a) and (b), respectively, we call this representation a factorization. A function that has singularities on the real axis may have more than one factorization. A simple example is

$$\bar{f}(\xi)=-\frac{1}{i\xi},\quad \bar{f}^+(\xi)=-\frac{A}{i\xi},\quad \bar{f}^-(\xi)=-\frac{1-A}{i\xi}$$

where A is any constant. Using (3.3), we obtain

$$f(x)=f_A(x)=AH(x)-(1-A)H(-x)$$

In particular, $f_0(x)=-H(-x)$, $f_1(x)=H(x)$. Consequently, for this function $\bar{f}(\xi)$ we have a whole class of locally integrable slowly-increasing functions, so that determination of the source function here involves taking its specific properties into account.

In particular, if $f(x)=0$ for $x>0$, then $\bar{f}_+(\xi)+0$; if $f(x)=0$ for $x<0$, then $\bar{f}_-(\xi)=0$.

Let us convolve these functions with $H(x)$:

$$g_+(x)=f_+(x)*H(x)=H(x)\int\limits_0^x f_+(y)\,dy,\quad g_-(x)=f_-(x)*H(-x)=$$

$$=H(-x)\int\limits_x^0 f_-(y)\,dy.$$

Using (3.1) and inverting the order of integration, we obtain

$$\bar{g}_{\pm}(\xi) = \lim_{\varepsilon\to+0} \frac{\bar{f}_+(\xi \pm i\varepsilon)}{\mp i(\xi \pm i\varepsilon)} \overset{\Delta}{=} \pm \frac{i\bar{f}_{\pm}(\xi \pm i0)}{\xi \pm i0} \tag{3.5}$$

Let $\bar{f}(\xi) = \bar{f}_-(\xi) + \bar{f}_+(\xi)$. Define the natural factorization of the function $\bar{f}(\xi)/i\xi$ to be the expression

$$\frac{\bar{f}(\xi)}{i\xi} = \frac{\bar{f}_+(\xi)}{i(\xi + i0)} + \frac{\bar{f}_-(\xi)}{i(\xi - i0)} \tag{3.6}$$

Using (3.5) we get

$$F_{\xi}^{-1}\left[\frac{\bar{f}(\xi)}{i\xi}\right] = f_+ * H(x) - f_- * H(-x) = H(x)\int_0^x f(y)\,dy - H(-x)\int_x^0 f(y)\,dy \tag{3.7}$$

4. GREEN'S TENSOR

Taking the Fourier transform of (2.1), we get an equation for the transform of Green's tensor $\bar{U}_{jk}^*(\xi)$, $\xi = (\xi_1, \xi_2, \xi_3)$:

$$(c_1^2 - c_2^2)\,\xi_i\xi_k\bar{U}_{kj}^* - (c_2^2\,\|\xi\|^2 - c^2\xi_3^2)\,\bar{U}_{ij}^* + \delta_{ij} = 0,$$

solution of which gives

$$\xi_k\bar{U}_{kj}^* = \xi_j(c_1^2\,\|\xi\|^2 - \mathrm{M}_1^2\xi_3^2)^{-1} \tag{4.1}$$

$$\bar{U}_{ij}^* = \delta_{ij}c_2^{-2}(\|\xi\|^2 - \mathrm{M}_2^2\xi_3^2)^{-1} + \frac{\xi_i\xi_j}{c^2\xi_3^2}\left((\|\xi\|^2 - \mathrm{M}_1^2\xi_3^2)^{-1} - (\|\xi\|^2 - \mathrm{M}_2^2\xi_3^2)^{-1}\right) \tag{4.2}$$

Depending on the value of M_j, the source matrix U_{ij} will have different forms.

Subsonic loads. Let $m_j = \sqrt{|1 - M_j^2|}$. If $c < c_j$ $(M_j < 1)$, then to determine U_{ij}^* we have to find the sources of the transforms

$$\bar{f}_{0j} = (\xi_1^2 + \xi_2^2 + m_j^2\xi_3^2)^{-1}$$

and their primitives as functions of x_3. Note that $\bar{f}_{0j}$ is the Fourier transform of a fundamental solution of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + m_j^2\frac{\partial^2}{\partial x_3^2}\right) f_{0j} + \delta(\mathbf{x}) = 0 \tag{4.3}$$

which transforms, by a change of variables $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3/m_j$, to the Laplace equation, whose Green's function is known [5]. The Fourier transform is

$$F_{y_1, y_2 y_3}[(4\pi R)^{-1}] = \|\xi\|^{-2},\quad R = \sqrt{y_1^2 + y_2^2 + y_3^2}$$

whence we obtain, by again changing variables,

$$f_{0j}(r, x_3) = (4\pi\sqrt{x_3^2 + m_j^2 r^2})^{-1},\quad r = \sqrt{x_1^2 + x_2^2}$$

The function $\bar{f}_{2j} = \bar{f}_{0j}/(i\xi_3)^2$ is the Fourier transform of the class of fundamental solutions of the equation

$$\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + m_j^2 \frac{\partial^2}{\partial x_3^2}\right)\Phi = -\delta(\mathbf{x}) \tag{4.4}$$

which can be expressed as $\Phi = \Phi_\delta + \Phi_0$, where Φ_δ is a particular fundamental solution of (4.4), Φ_0 a solution of the homogeneous equation corresponding to (4.4). To determine Φ_δ, we use the natural factorization (3.7) of $\bar{f}_{0j}(\xi)/(i\xi_3)$ as a function of ξ_3. Define

$$\Phi_{kj}(r, x_3) = H(x_3)\int_0^{x_3} \Phi_{(k-1)j}(r, y)\,dy - H(-x_3)\int_{x_3}^{0} \Phi_{(k-1)j}(r, y)\,dy, \quad k = 1, 2$$

$$\Phi_{1j} = (4\pi)^{-1}\operatorname{sgn} x_3 \ln\left(\frac{|x_3| + V_j^+}{m_j r}\right), \quad V_j^\pm = \sqrt{x_3^2 \pm m_j^2 r^2};$$

$$\Phi_{2j} = (4\pi)^{-1}\left(|x_3| \ln\left(\frac{|x_3| + V_j^+}{m_j r}\right) - V_j^+ + m_j r\right)$$

$$f_{kj} = \Phi_{kj},\ k = 0, 1;\ f_{2j} = \Phi_{2j} - (4\pi)^{-1} m_j r$$

By (4.3), the function Φ_{2j} satisfies Eq. (4.4). Since the term $m_j r$ is a solution of the corresponding homogeneous equation, $f_{2j}(r, x_3)$ is also a fundamental solution of the same equation. We shall use it to construct U_{ij}^*.

It follows from (4.1) that

$$U_{ij,i}^* = c_1^{-2} f_{01,j}$$

$$U_{ij}^* = c_2^{-2}\delta_{ij} f_{02}(r, x_3) + c^{-2}\left(f_{21}(r, x_3) - f_{22}(r, x_3)\right)_{,ij}$$

Differentiating, we obtain formulae identical with those obtained in [2] by direct inversion of the transforms of the solutions:

$$U_{11}^* = \frac{1}{4\pi c_2^2}\left[\frac{1}{V_2^+} + \frac{x_3^2 x_1^2}{r^4 M_2^2}\left(\frac{v}{V_1^+} - \frac{1}{V_2^+}\right) - \frac{x_2^2}{M_2^2 r^4}(V_1^+ - V_2^+)\right]$$

$$U_{22}^* = \frac{1}{4\pi c_2^2}\left[\frac{1}{V_2^+} + \frac{x_3^2 x_2^2}{r^4 M_2^2}\left(\frac{1}{V_1^+} - \frac{1}{V_2^+}\right) - \frac{x_1^2}{M_2^2 r^4}(V_1^+ - V_2^+)\right]$$

$$U_{33}^* = \frac{1}{4\pi c^2}\left(\frac{1}{V_1^+} - m_2^2 \frac{1}{V_2^+}\right)$$

$$U_{j3}^* = -\frac{x_j x_3}{4\pi c^2 r^2}\left(\frac{1}{V_1^+} - \frac{1}{V_2^+}\right), \quad j = 1, 2$$

$$U_{12}^* = \frac{x_1 x_2}{4\pi c^2 r^4}\left[(V_1^+ - V_2^+) + x_3^2\left(\frac{1}{V_1^+} - \frac{1}{V_2^+}\right)\right]$$

Since when $r \to 0$, $x_3 \neq 0$,

$$V_1^+ - V_2^+ \sim 0{,}5 r^2 (m_1^2 - m_2^2)|x_3|^{-1}, \quad \left(\frac{1}{V_1^+} - \frac{1}{V_2^+}\right) \sim \frac{r^2 (m_1^2 - m_2^2)}{2|x_3|^3}$$

$$\frac{x_1 x_3}{r^4}\left(\frac{1}{V_1^+} - \frac{1}{V_2^+}\right) - \frac{x_2^2}{r^4}(V_1^+ - V_2^+) \sim \frac{1}{2|x_3|}(m_1^2 - m_2^2)$$

and so the tensor U_{ij}^* has removable singularities on the line $r = 0$, in addition to the point $x = 0$. As $R \to 0$ we have $U_{ij}^* \sim \mathrm{const}/R$, $T_{ij}^* \sim \mathrm{const}/R^2$, $R = \|x\|$, and it can be shown that the integrals in

(2.9) either exist for $\mathbf{x}\in S_+$ or exist in the principal-value sense for $\mathbf{x}\in S$. The construction of the BIE in this case proceeds as in the static theory of elasticity and can be accomplished by going to the limit in (2.9).

Supersonic loads. If $c>c_j$ $(M_j>1)$ we must determine the sources of the transforms

$$\bar{g}_{0j} = (\xi_1^2 + \xi_2^2 - m_j^2\xi_3^2)^{-1}$$

Clearly, the support of g_{0j} must lie on the negative half-axis $x_3<0$, since the load "outstrips" the perturbations propagating in the medium. Using the fundamental solution of the wave equation [5], we obtain

$$g_{0j}(r, x_3) = \frac{H(-x_3 - m_j r)}{2\pi V_j^-}, \quad g_{1j} = -H(-x_3)\int_{x_3}^{0} g_{0j}(r, y)\,dy =$$

$$= -\frac{H(-x_3 - m_j r)}{2\pi}\ln\left(\frac{|x_3| + V_j^-}{m_j r}\right), \quad g_{2j} = -H(-x_3)\times$$

$$\times\int_{x_3}^{0} g_{1j}(r, y)\,dy = -\frac{H(-x_3 - m_j r)}{2\pi}\left(|x_3|\ln\left(\frac{|x_3| + V_j^-}{m_j r}\right) - V_j^-\right)$$

Hence, in the intermediate (transonic) range $(c_2<c<c_1)$:

$$U_{ij}^* = \delta_{ij}c_2^{-2}g_{02}(r, x_3) + c^{-2}(f_{21}(r, x_3) - g_{22}(r, x_3))_{,ij} \tag{4.5}$$

For supersonic velocities $(c>c_1)$ we must replace f_{21} in (4.5) by g_{21}.

As follows from the formulae, the fronts obtained in these cases are conical surfaces: $|x_3| = m_j r$, $x_3<0$, $j = 1, 2$.

At transonic velocities there is just one such front, downstream from which there propagates only a body deformation, while upstream there is also a shear deformation. At supersonic velocities $(c>c_1)$ there are two fronts; the medium downstream from the leading front is at rest. The fronts propagate at velocities c_1, c_2, respectively.

Sonic loads. If $c = c_j$ we have $m_j = 0$, and then

$$h_0(r, x_3) = F^{-1}_{\xi_1\xi_2\xi_3}[(\xi_1^2 + \xi_2^2)^{-1}] = -(2\pi)^{-1}\delta(x_3)\ln r$$

Here we have used the fundamental solution of the two-dimensional Laplace equation [5]. Similarly, following (3.7), we obtain

$$h_1(r, x_3) = (2\pi)^{-1}\delta(x_3)\ln r *_{x_3} H(-x_3) = (2\pi)^{-1}H(-x_3)\ln r$$

$$h_2(r, x_3) = -(2\pi)^{-1}H(-x_3)\ln r *_{x_3} H(-x_3) = (2\pi)^{-1}x_3 H(-x_3)\ln r$$

If $c = c_2$:

$$U_{ij}^* = c_2^{-2}\delta_{ij}h_0(r, x_3) + c^{-2}(f_{21}(r, x_3) - h_2(r, x_3))_{,ij}$$

If $c = c_1$:

$$U_{ij}^* = c_2^{-2}\delta_{ij}g_{02}(r, x_3) + c^{-2}(h_2(r, x_3) - g_{22}(r, x_3))_{,ij}$$

At sonic velocities, the wave front corresponding to $c = c_j$ is perpendicular to the x_3 axis and coincides with the x_1x_2 plane in the moving coordinate frame.

To derive an integral analogue of the Somigliana formula and a BIE in the supersonic case, one has to use formulae (2.7), since the integrals in (2.9) fail to exist not only on the boundary ($\mathbf{x} \in S$) but even inside the domain ($\mathbf{x} \in S_+$). This arises from the existence of non-integrable singularities of the kernel T_{ij} on the fronts, implying the need for non-trivial regularization of the integrals in (2.7) before differentiation.

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Translated by D.L.